Statistics C206B Lecture 2 Notes

Daniel Raban

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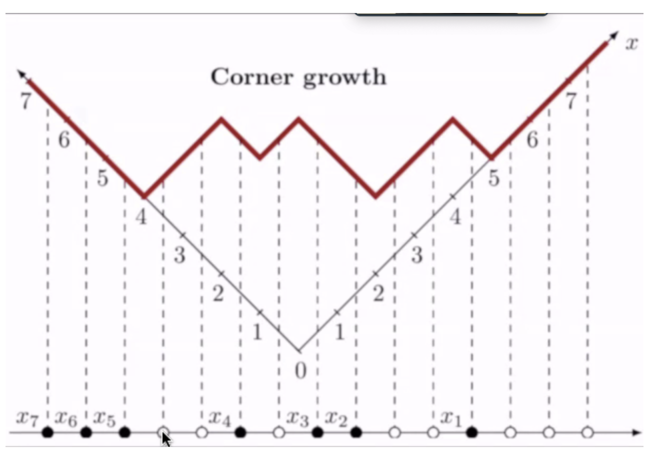
1 More Examples of Random Interfaces

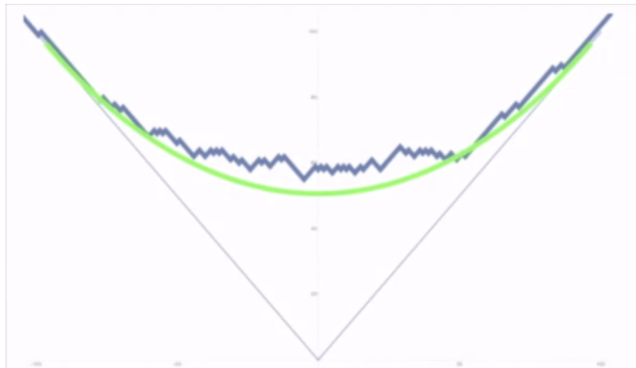
1.1 Growth models

The goal for today is to continue discussion of interesting examples and relevant questions in 1 dimension and higher dimensions. We will also do a quick review of basic Markov chain theory, mixing of Markov chains, and some general tools useful to bound mixing times.

Recall the corner growth model from last time.

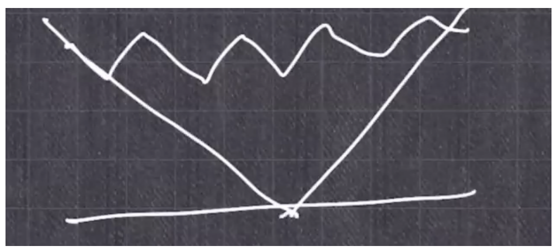


A simulation of this curve shows that with large n, it looks like a parabola with random (non-Gaussian) fluctuations.

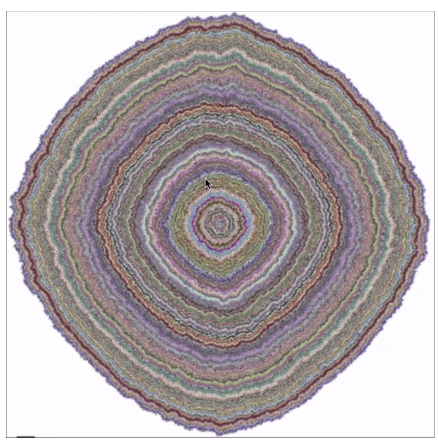


In fact, this will be related to random matrices and the Tracy-Widom distribution

In many models of random growth, the relevant interface will *not* be a function.



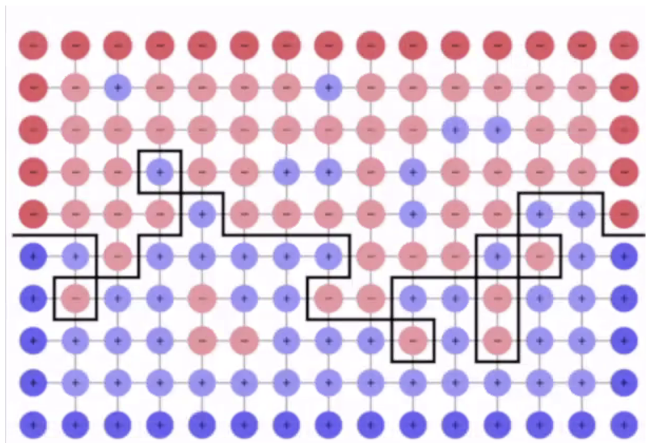
Example 1.1. The **Richardson model** is a random growth model on $\mathbb{Z}^2$. Here is how we construct it: Start with the origin. Sites on $\mathbb{Z}^2$ get attached to the existing growth cluster at the rate of the number of neighbors already in the cluster. Here is a picture of the cluster, colored by when the sites were added.



The boundary is rough and determined randomly.

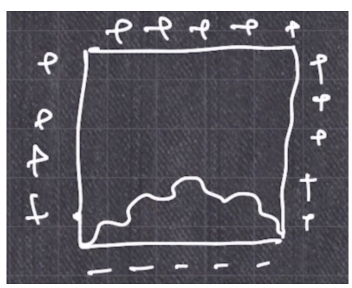
1.2 Ising model with magnetic fields

Example 1.2. Recall the Ising model, with state space $\{\pm 1\}^\Lambda$ and measure $P(\sigma) \propto \exp\left(-\beta \sum_{u\sim v} \mathbb{1}_{\{\sigma_u \neq \sigma_v\}}\right)$. If we add a magnetic field, we add an extra term $\lambda \sum_u \sigma_u$ inside the exponent. We mentioned **Dobrushin's boundary condition**, where one half of the boundary is assigned $+$ and the other half is assigned $-$. If the temperature is low enough ($\beta > \beta_c$), then two phases will coexist. Here is an example:



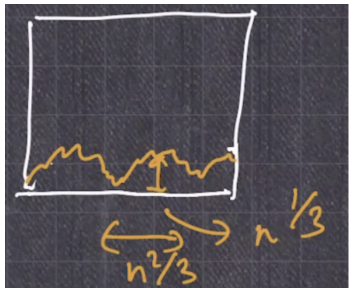
Here, we have bubbles and overhangs which show why we may not get a function. It is known that the interface scales to a Brownian bridge. However, adding a magnetic field changes the behavior.

Suppose 3 sides of the boundary are $+$ and 1 side of the boundary is $-$.



Under proper scaling, this converges to a **Brownian excursion** (Brownian bridge conditioned on remaining nonnegative). Now suppose we add a field with $\lambda = O(1)$; if $\lambda > 0$, then the system is more likely to be positive, pushing down the interface. In the case fo $\lambda = O(1)$, the fluctuations of the interface will have height $O(1)$; this is called **partially wetted** regime.

The interesting regime is when $\lambda \to 0$ with n, called the **prewetting regime**. If $\lambda = 1/n$, it turns out that the interface will look like Brownian excusions on windows of size $n^{2/3}$ with height $n^{1/3}$.

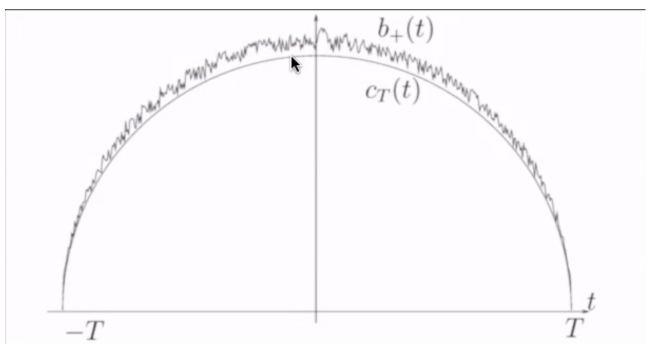


This lives in the KPZ universality class.

Here is a proxy model for this type of behavior.

Example 1.3 (Area tilted Brownian polymers)**.** Let μ be the Brownian excursion measure. Then the measure π is given by $\frac{d\pi}{d\mu}(\gamma) = e^{-\lambda \operatorname{Area}(\gamma)}$, which penalizes curves that have too much area beneath them.

Example 1.4. Another example is a Brownian bridge conditioned to stay above a given function.



Here, the curve will be penalized for having more area under it that is not under the semicircle.

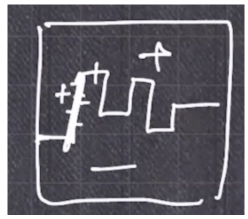
1.3 Solid on solid models

Here is how we can approximate the Ising model behavior by a function:

Example 1.5 (Solid on solid models)**.** This will be a measure on functions $f : \{0, \dots, n\} \to \mathbb{Z}$ or to $\mathbb{Z}_+$ (depending on if we want to approximate a Brownian bridge or a Brownian excursion). Assume $f(0) = 0 = f(n)$. The measure is

$$\begin{aligned} P(f) &\propto e^{-\beta \sum_i |f(i)-f(i-1)|} \\ &= \prod_i e^{-\beta |f(i)-f(i-1)|}, \end{aligned}$$

which discourages large changes in the function, which correspond to large boundaries with one side $+$ and another side $-$.



If we write $X_i = f(i) - f(i-1)$, we can think of the SOS model as the random walk with the above increment distribution, conditioned to be 0 at n.

Here is a generalization.

Example 1.6 (Ginzburg Landau $\nabla\phi$ models)**.** This is a class of measures on functions given by

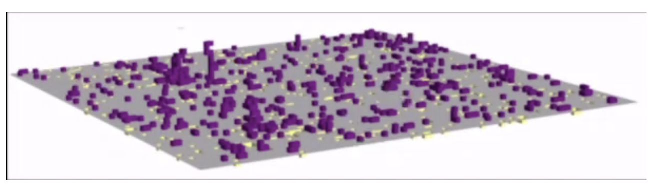
$$P(f) \propto e^{-\beta \sum_i \phi(f_i - f_{i-1})}.$$

The most studied example is $\phi(x) = x^2$, which corresponds to a Gaussian random walk.

We will study the naturally associated Glauber dynamics, which resamples the value of the function at a given site according to the conditional measure.

1.4 2 dimensional Ising interface and the Gaussian free field

Example 1.7. We can set up the Ising model in a 3-dimensional box with the top half of the boundary being $+$ and the bottom half being $-$. The interface will be a surface.

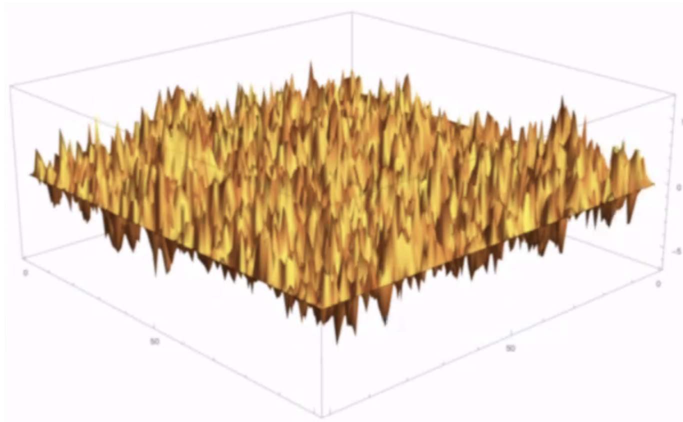


We can also study SOS and Ginzburg-Landau models in this setting:

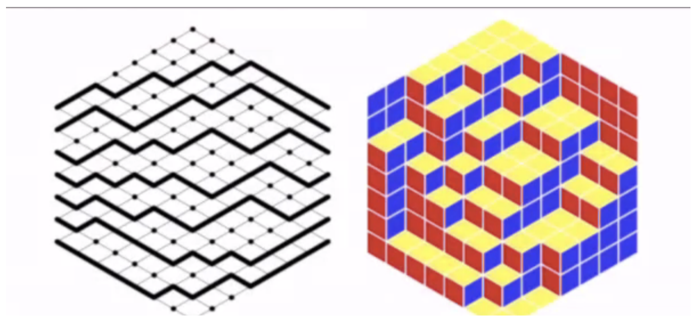
Example 1.8. The **Gaussian free field** is the G-L model measuring functions $f : [0,n]^2 \to \mathbb{R}$ with

$$\mu(f) \propto e^{-\sum_{u \sim v} (f(u) - f(v))^2}.$$

This is an analogue of Brownian motion in higher dimensions and appears as the limiting distribution in many central examples in statistical mechanics. Here is a picture of a simulation of the GFF in 2 dimensions:



Example 1.9. There is a class of random tilings of the plane which gives rise to height functions related to the Gaussian free field. Here, if we tile the plane with 3 colors, we can construct a height function by viewing the tiling as boxes stacked on top of each other.



We can also think of these tilings in terms of non-intersecting lattice paths.